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LETTER TO THE EDITOR

On universality of the smoothed eigenvalue density of large random matrices

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Abstract. We describe the resolvent approach for the rigorous study of the mesoscopic regime of Hermitian matrix spectra. We present results reflecting universal behaviour of the smoothed density of the eigenvalue distribution of large random matrices.

Random matrices of large dimensions introduced and studied by Wigner [1] have applications in various fields of theoretical physics (see e.g. monographs and reviews [2–4] and references therein). In these studies, the spectral properties of random matrix ensembles play an important role.

Here the universality conjecture for large random matrices, formulated by Dyson [2, 5], is known as the most interesting and challenging problem. It concerns the asymptotically local spectral statistics, i.e. the functions that depend on a certain number, q, of eigenvalues of the random $N \times N$ matrix A_N, where q remains fixed when $N \to \infty$.

Loosely speaking, the universality conjecture states that the local statistics regarded in the limit $N \to \infty$, do not depend on the details of the probability distribution $P(A_N)$ of the ensemble but are determined by the symmetries of the ensemble. For example, the expressions derived for local statistics of Hermitian ensembles are different from those of real symmetric matrices.

Given a Hermitian (or real symmetric) matrix A_N, the distribution of its eigenvalues $\lambda_1^{(N)} \leqslant \cdots \leqslant \lambda_N^{(N)}$ is determined by the normalized eigenvalue counting function

$$\sigma_N(\lambda) \equiv \sigma(\lambda; A_N) := \#\{\lambda_j^{(N)} \leqslant \lambda\} N^{-1}$$

or, equivalently, by the associated measure

$$\sigma_N(\Delta) \equiv \int_a^b \rho_N(\lambda)\, \mathrm{d}\lambda \qquad \Delta = (a, b) \subset \boldsymbol{R}$$

with the formal density

$$\varrho_N(\lambda) = \frac{1}{N} \sum_{j=1}^{N} \delta(\lambda - \lambda_j^{(N)}). \tag{1}$$

The function $\sigma_N(\lambda)$ is called the empirical eigenvalue distribution function. Regarding $\sigma_N(\Delta_N)$, it turns out to be the local spectral statistics when considered with the intervals of the length $|\Delta_N| = \mathrm{O}(1/N)$ as $N \to \infty$.

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In general, the local spectral regime is rather hard to rigorously analyse. The universality conjecture is supported mainly for those ensembles of random matrices that have the explicit form of the joint probability distribution $\pi_N(\lambda_1, \ldots, \lambda_N)$ of eigenvalues. Starting from π_N, the same expression for an m-point correlation function is derived by Dyson for the circular ensemble of unitary random matrices (CUE) [2], by Mehta for GUE [6], and by Pastur and Shcherbina for the matrix models ensemble [7]. This expression is given by the determinant of an $m \times m$ matrix with the entries $\{\sin \pi(t_i - t_j)/\pi(t_i - t_j)\}$, $i, j = 1, \ldots, m$. The same expression is derived in [8] for a random matrix ensemble with the entries that are independent random variables, whose probability distribution is a convolution of the Gaussian distribution and the arbitrary one.

Our principal goal is to examine the presence of universality of the spectral characteristics for those ensembles of random matrices, for which the explicit form of the joint eigenvalue distribution π_N is unknown. For example, the random matrix with independent ± 1 entries falls into this class. Our claim is that the eigenvalue density (1) smoothed over the intervals $\Delta_N \subset \boldsymbol{R}$ possesses the universal properties as $N \to \infty$ provided the length $l_N = |\Delta_N|$ satisfies the conditions $1 \ll l_N \ll N$.

We determine the smoothing (or regularization) of (1) by the formula

$$R_N^{(\alpha)}(\lambda) := \int_{-\infty}^{\infty} \frac{N^\alpha}{1 + N^{2\alpha}(\lambda - \lambda')^2} \varrho_N(\lambda')\, \mathrm{d}\lambda' \tag{2}$$

and note that in this case

$$R_N^{(\alpha)}(\lambda) = \mathrm{Im}\ \mathrm{Tr}\, G_N(\lambda + \mathrm{i}N^{-\alpha})N^{-1}$$

where $G_N(z) = (A_N - z)^{-1}$.

According to the above definition, $\xi_N^{(\alpha)}(\lambda)$ with $\alpha = 1$ represents the asymptotically local spectral statistics. The opposite asymptotic regime when $\alpha = 0$ is known as the global one. In this case the limit

$$g(z) = \lim_{N\to\infty} \mathrm{Tr}\, G_N(z)N^{-1} \qquad |\mathrm{Im}\, z| > 0$$

if it exists, determines the limiting eigenvalue distribution $\sigma(\lambda)$ of the ensemble $\{A_N\}$; that is

$$\sigma(\lambda) = \lim_{N\to\infty} \sigma_N(\lambda) \qquad g(z) = \int_{-\infty}^{\infty} (\lambda - z)^{-1}\, \mathrm{d}\sigma(\lambda).$$

Regarding the global regime, the resolvent approach developed in [9, 10] is proved to be rather effective in studies of the eigenvalue distribution of large random matrices (see, for example [11–13]). In this regime the limit of $g_N(z) = \mathrm{Tr}\, G_N(z)N^{-1}$ depends on the probability distribution of the ensemble, i.e. is non-universal, as well as the fluctuations of $g_N(z)$ [11, 12].

We are interested in the behaviour of (2) in the case of $0 < \alpha < 1$. This regime is intermediate between the local and the global ones. It can be called the mesoscopic regime in random matrix spectra. For this regime, the modified version of the resolvent approach was proposed in [14] to study spectral properties of random matrices with independent arbitrary distributed entries (see also [15, 16]).

As a further development of the resolvent approach of [14], we present the results concerning random matrices with statistically dependent entries. We consider the ensemble of random matrices

$$H_{m,N}(x, y) = \frac{1}{N} \sum_{\mu=1}^{m} \xi_\mu(x)\xi_\mu(y) \qquad x, y = 1, \ldots, N \tag{3}$$

where the random variables $\{\xi_\mu(x)\}$, $x, \mu \in \boldsymbol{N}$ have joint Gaussian distribution with zero mathematical expectation and covariance

$$\boldsymbol{E}\{\xi_\mu(x)\xi_\nu(y)\} = u^2\delta_{xy}\delta_{\mu\nu}.$$

Here δ_{xy} denotes the Kronecker delta-symbol. This ensemble, first considered in [17], is now of extensive use in the statistical mechanics of disordered spin systems [4] and in the modelling of memory in the theory of neural networks [18].

Theorem 1. *Let $G_{m,N}(z) = (H_{m,N} - z)^{-1}$. Then, for $N, m \to \infty$, $m/N \to c > 0$, the random variable*

$$R^{(\alpha)}_{m,N}(\lambda) := \mathrm{Im}\ \mathrm{Tr}\, G_{m,N}(\lambda + \mathrm{i}N^{-\alpha})N^{-1}$$

converges with probability 1 to the nonrandom limit

$$\pi\varrho_c(\lambda) = \frac{1}{2\lambda u^2}\left(4cu^4 - [\lambda - (1 + cu^2)]^2\right)^{1/2} \tag{4}$$

provided $0 < \alpha < 1$ *and* $\lambda \in \Lambda_{c,u} = (u^2(1-\sqrt{c})^2, u^2(1+\sqrt{c})^2)$.

Theorem 2. *Consider k random variables*

$$\gamma^{(\alpha)}_{m,N}(i) := N^{1-\alpha}[R^{(\alpha)}_{m,N}(\lambda_i) - \boldsymbol{E}R^{(\alpha)}_{m,N}(\lambda_i)] \qquad i = 1, \ldots, k$$

where $\lambda_i = \lambda + \tau_i N^{-\alpha}$ with given τ_i. Then, under the conditions of theorem 1 the joint distribution of the vector $(\gamma_N(1), \ldots, \gamma_N(k))$ converges to the Gaussian k-dimensional distribution with zero average and covariance

$$C(\tau_i, \tau_j) = \frac{4 - (\tau_i - \tau_j)^2}{[4 + (\tau_i - \tau_j)^2]^2}. \tag{5}$$

Remark. It is easy to see that if $|\tau_1 - \tau_2| \to \infty$, then

$$C(\tau_1, \tau_2) = -(\tau_1 - \tau_2)^{-2}(1 + \mathrm{o}(1)). \tag{6}$$

This coincides with the average value of Dyson's 2-point correlation function for real symmetric matrices considered at large distances $|t_1 - t_2| \gg 1$ [2].

To discuss these results, let us first note that theorem 1 proves the existence of the smoothed density of eigenvalues that coincides with that derived in [17] in the global regime; $\varrho_c(\lambda) = \sigma'_c(\lambda)$, $\lambda > 0$, where

$$\sigma_c(\lambda) = \lim_{N\to\infty} \sigma(\lambda; H_{m,N}).$$

This density obviously differs from the semicircle (or Wigner) distribution $\sigma_w(\lambda)$

$$\sigma_w(\lambda) = \lim_{N\to\infty} \sigma(\lambda; W_N)$$

where $W_N(x, y) = w(x, y)/\sqrt{N}$ are random symmetric matrices with independent identically distributed entries with zero mathematical expectation and variance v^2. This ensemble is known as the Wigner ensemble of random matrices. It has been known since the pioneering work of Wigner [1] that

$$\varrho_w(\lambda) = \sigma'_w(\lambda) = \frac{1}{2\pi v^2}\begin{cases} (4v^2 - \lambda^2)^{1/2} & \text{if} \quad |\lambda| \leqslant 2v \\ 0 & \text{if} \quad |\lambda| > 2v. \end{cases}$$

It should be noted that in [14, 16] we proved analogues of theorems 1 and 2 for the Wigner ensemble of random matrices. We have shown that theorem 2 is true provided $\boldsymbol{E}w(x, y)^8$ is bounded and $\alpha \in (0, \frac{1}{8})$. The correlation function $C(\tau_i, \tau_j)$ is given again by (5). Comparing these results, we conclude the fluctuations of the smoothed eigenvalue density do not feel the dependence between matrix elements.

Thus, our results can be regarded as statements corroborating the universality conjecture for the mesoscopic regime. Namely, they show that in the mesoscopic regime the smoothed density of eigenvalues $R_N^{(\alpha)}$ is the self-averaging variable. It converges, as $N \to \infty$, to the eigenvalue distribution of the ensemble and depends on the probability distribution of the random matrix ensemble. At the same time the fluctuations of $R_N^{(\alpha)}$ in the limit $N \to \infty$ coincide for two such different classes of random matrices as (3) and the Wigner one.

This dual-type behaviour of the eigenvalue density of random matrices in the mesoscopic regime is well known in theoretical physics (see, for example, the review [3], ch 8). For example, the universal properties of the mesoscopic eigenvalue density were studied in [19,20] for the matrix models ensemble. It was shown that the correlation function of the eigenvalue density (in theoretical physics terms, the 'wide' correlator) depends on the edges of the spectrum. For the case of the symmetric support of the limiting eigenvalue distribution, the expression for the 'wide' correlator coincides with the asymptotic expression (6). It should be noted that our result (6) does not depend on the support of $\varrho_c(\lambda)$.

Let us describe the method developed for the proof of theorems 1 and 2 (the full version will be published elsewhere). It represents a modification of the resolvent approach proposed in [9,10]. This approach was developed to study the eigenvalue distribution of random matrices and random operators in the global regime $|\mathrm{Im}\, z| > 0$ as $N \to \infty$. It is based on the derivation and asymptotic analysis of the system of relations for the moments $L_k^{(N)} = \boldsymbol{E}[g_N(z)]^k$, $k \geqslant 1$. These relations are of the following form:

$$L_k = aL_{k-1} + bL_{k+1} + \Phi_k^{(N)} \tag{7}$$

where the terms $\Phi_k^{(N)}$ can be estimated by $N^{-1}|\mathrm{Im}\, z|^{-k}$. This *a priori* estimate implies that in the study of the asymptotic behaviour of $g_N(z)$, one can restrict oneself to only the first two relations. Namely, all information about the limiting behaviour of $L_k^{(N)}$, $k \geqslant 1$ can be derived from relations for $L_1^{(N)}$ and $L_2^{(N)}$.

To consider $g_N(z)$ in the mesoscopic regime, we start with the same system of relations for $L_k^{(N)}$. The main observation made in [14] is that in this case we need all the infinite system of relations. More precisely, the closer α is to 1, the greater number $K(\alpha)$ becomes, such that we need to consider relations for $L_k^{(N)}$, $k \geqslant K(\alpha)$.

The matter is that the terms $\Phi_k^{(N)}$ can be estimated in terms of $L_k^{(N)}$ multiplied by $N^{-\beta}$, where $\beta = \min\{\alpha, 1-\alpha\}$. The structure of relation (7) is such that $L_j^{(N)}$, $j < k$ enter into the relation for $L_k^{(N)}$ with the factor $N^{\beta(k-j)}$. Therefore, admitting *a priori* estimate $|L_1^{(N)}| \leqslant N^{\alpha}$, we deduce that it enters into relation (7) with the factor $N^{-k\beta}$. Regarding, $k \geqslant K(\alpha)$, one obtains the relations with the terms that converge to finite limits as $N \to \infty$.

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References

[1] Wigner E 1955 *Ann. Math.* **62** 548–64
[2] Dyson F 1962 *J. Math. Phys.* **3** 140–75
[3] Guhr T, Müller-Groeling A and Weidenmüller H A 1998 *Phys. Rep.* **299**
[4] Mézard M, Parisi G and Virasoro M 1987 *Spin Glass Theory and Beyond* (Singapore: World Scientific)

[5] Dyson F 1972 *J. Math. Phys.* **13** 90
[6] Mehta M L 1991 *Random Matrices* (New York: Academic)
[7] Pastur L and Shcherbina M 1997 *J. Stat. Phys.* **86** 109–47
[8] Johansson K 1998 *Preprint*
[9] Khorunzhy A and Pastur L 1993 *Commun. Math. Phys.* **153** 605–46
[10] Khorunzhy A and Pastur L 1994 Spectral operator theory and related topics *Adv. Sov. Math.* **19** 97–127
[11] Boutet de Monvel A and Khorunzhy A 1998 *Markov Proc. Relat. Fields* **4** 175–97
[12] Khorunzhy A, Khoruzhenko B and Pastur L 1996 *J. Math. Phys.* **37** 5033–60
[13] Khorunzhy A and Rodgers G J 1998 *Rep. Math. Phys.* **43** 297–319
[14] Khorunzhy A 1997 *Random Oper. Stoch. Eqns* **5** 147–62
[15] Boutet de Monvel A and Khorunzhy A 1999 *Random Oper. Stoch. Eqns* **7** 1–22
[16] Boutet de Monvel A and Khorunzhy A 1999 *Random Oper. Stoch. Eqns* **7** to appear
[17] Marchenko V and Pastur L 1967 *Math. USSR Sbornik* **1** 457–83
[18] Hertz J A, Krogh A and Palmer R G 1991 *Introduction to the Theory of Neural Computations* (Reading, MA: Addison-Wesley)
[19] Ambjørn J, Jurkiewicz J and Makeenko Yu M 1990 *Phys. Lett.* B **251** 517
[20] Brézin E and Zee A 1993 *Nucl. Phys.* B **402** 613–27